Schur polynomials and GL(n)

IF λ IS A PARTITION $\lambda \vdash k$

$\lambda = (\lambda_1, \cdots, \lambda_r)$ $\quad \lambda_1 \geq \lambda_2 \geq \cdots$

$\sum \lambda_i = k$ THERE IS A POLYNOMIAL

$$D_\lambda(x_1, \cdots, x_n) = \frac{\det(x_j^{\lambda_i + n - i})}{\det(x_j^{n-i})}$$

ANY $n \geq r$

$n = 3 \qquad \lambda = (2,1)$

$$\frac{\begin{vmatrix} x_1^{\lambda_1+2} & x_2^{\lambda_1+2} & x_3^{\lambda_1+2} \\ x_2^{\lambda_2+1} & x_2^{\lambda_2+1} & x_3^{\lambda_2+1} \\ x_1^{\lambda_3} & x_2^{\lambda_3} & x_3^{\lambda_3} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{vmatrix}}$$

$$= \begin{vmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{vmatrix} \Big/ \begin{array}{l} (x_1 - x_2)(x_1 - x_3) \\ (x_2 - x_3) \end{array}$$

VANDERMONDE: $\det\left(x_j^{n-i}\right) = \prod_{i<j} (x_i - x_j)$

TO SEE THIS IS A POLYNOMIAL, NOTE NUMERATOR IS ANTISYMMETRIC;

INTERCHANGING X_1 AND X_2 CHANGES SIGN SO THIS POLYNOMIAL IS DIV. Bx $X_1 - X_2$, SIMILARLY OTHER FACTORS $x_i - x_j$ WHICH ARE COPRIME IN

$R[x_1, \ldots, x_n]$.

<u>THEOREM</u>: IF λ IS A PARTITION OF n THERE EXISTS AN IRREDUCIBLE REP'N OF $GL(n, \mathbb{C})$ CALL IT π_λ WITH CHAR. $\chi_\lambda(g) = \mathrm{tr}\, \pi_\lambda(g)$ SATISFIES

$$\chi_\lambda(g) = s_\lambda(\alpha_1, \dots, \alpha_n)$$

WHERE α_i ARE EIGENVALS OF g.

SO

$$\pi_\lambda : GL(n, \mathbb{C}) \longrightarrow GL(m, \mathbb{C})$$

IS A HOMOMORPHISM

WHERE $m = s_\lambda(1, \dots, 1)$

TAKING THE TRACE $\chi_\lambda(g) = \mathrm{tr}\, \pi_\lambda(g)$

PRODUCES THE <u>CHARACTER</u>.

EXAMPLE: $\lambda = (0, \dots, 0)$

$\Delta_\lambda(x_1, \dots, x_n) = 1$

REP'N $\pi_\lambda : GL(n, \mathbb{C}) \to GL(1, \mathbb{C})$

$\pi_\lambda(g) = 1$ FOR ALL g

TRIVIAL REPRESENTATION.

$\lambda = (k, 0, 0, \dots)$

$\Delta_{(k, 0, \dots, 0)}(x_1, \dots, x_n)$ IS THE

"COMPLETE" SYMMETRIC POLYNOMIAL

DENOTED h_k

$$h_k(x_1, \dots, x_n) = \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k}$$

THIS IS THE SUM OF ALL MONOMIALS OF DEGREE k.

THIS IS THE CHARACTER OF A REPRESENTATION THE k-TH SYMMETRIC POWER REPRESENTATION

EQUIVALENT BUT MAY SKIP.

$P_k(\mathbb{C}^n)$ = ALL POLYNOMIALS HOMOGENEOUS OF DEGREE k

$\mathbb{C}^n = V$ COMPLEX VECTOR SPACE

$$P_k(V^*) = \mathrm{Sym}_k(V) = \vee^k V$$

HOMOGENEOUS PART OF DEGREE k IN THE "SYMMETRIC ALGEBRA".

$$\bigoplus \mathrm{Sym}_k(V) = \mathrm{Sym}(V)$$

SYMMETRIC ALGEBRA

= RING OF POLYNOMIALS ON V^*.

GL(n) ACTS ON V^*. $f \in V^*$

$(gf)(v) = f(g^{-1}v)$ $f: V \to \mathbb{C}$ IS A LINEAR FUNCTIONAL

GIVEN A F.D. VECTOR SPACE V THERE ARE 3 ALGEBRAS WE CAN DEFINE 2 ARE ∞ DIM'L, ONE (THE EXTERIOR ALGEBRA) IS FINITE-DIM'L.

I TENSOR ALGEBRA.

IF $\dim(V) = n$ LET $v_1, \ldots, v_n$ BE A BASIS

$$\underbrace{V \otimes \cdots \otimes V}_{k \text{ FACTORS}} = \otimes^k V$$

SPANNED BY $v_{i_1} \otimes \cdots \otimes v_{i_k}$

SO $\dim\left(\otimes^k V\right) = n^k$

$\otimes^0 V = \mathbb{C}$

$\bigoplus_{k=0}^{\infty} \left(\otimes^k V\right)$ IS AN GRADED ALGEBRA

WITH MULTIPLICATION

$$\left(\otimes^k V\right) \times \left(\otimes^l V\right) \to \otimes^{k+l} V$$

$$(a, b) \longmapsto a \otimes b$$

THIS HAS TWO QUOTIENTS.

SYMMETRIC ALGEBRA

SYMMETRIC GROUP S_n ACTS

ON $\bigotimes^n V$ BY $\sigma \in S_n$

$$\sigma(u_1 \otimes \cdots \otimes u_n) =$$

$$u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(n)}.$$

$\sigma(\tau \xi) = (\sigma\tau)(\xi)$

K = SUBSPACE SPANNED BY

$$a - \sigma(a) \qquad \sigma \in S_n, \ a \in \bigotimes^n V.$$

THE QUOTIENT $\bigotimes^n V / K := \vee^n V$

IS THE n-TH SYMMETRIC POWER.

($\vee$, $\wedge$ SPECIAL SYMBOLS \vee, \wedge)

DENOTE THE IMAGE OF

$$u_1 \otimes \cdots \otimes u_k \quad \text{IN} \quad \vee^k V$$

BY $u_1 \vee \cdots \vee u_k$

UNCHANGED BY PERMUTATIONS

OF $u_1, \cdots, u_k$.

$k = 2 \qquad n = 4$

BASIS OF V IS $v_1, \cdots, v_4$

BASIS OF $\vee^2 V$ IS

$v_1 \vee v_1, \; v_1 \vee v_2, \; v_1 \vee v_3, \; v_1 \vee v_4$

$v_3 \vee v_3 \quad v_2 \vee v_4, \; v_3 \vee v_4$

$v_2 \vee v_2 \quad v_3 \vee v_3 \quad v_4 \vee v_3$

$$\binom{n+k-1}{k} = \binom{5}{2} = 10.$$

BASIS IS

$$v_{i_1} \vee v_{i_2} \vee \cdots \vee v_{i_k}$$

$$i_1 \leq i_2 \leq \cdots \leq i_k$$

$GL(n, \mathbb{C})$ ACTS ON $V = \mathbb{C}^n$

HENCE ON $\otimes^k \mathbb{C}^n$

AND ALSO ON $\vee^k \mathbb{C}$.

WE HAVE REPRESENTATIONS

$$GL(n,\mathbb{C}) \to GL(W)$$

W ANY OF $\otimes^k V$, $\vee^k V$

OR $\wedge^k V$ ← NOT DEFINED YET

BUT DEFINED SIMILARLY.

IF THE EIGENVALUES OF g

ARE $\alpha_1, \dots, \alpha_n$

E.G. $g = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix}$

$$g : v_i \to \alpha_i v_i$$

$$g : v_{i_1} \wedge \cdots \wedge v_{i_n} \to$$

$$\underbrace{\alpha_{i_1} \cdots \alpha_{i_n}}\, v_{i_1} \wedge \cdots \wedge v_{i_n}$$

EIGENVALUE OF MATRIX $\wedge^n g$ INDUCED BY g IS ACTION ON $\wedge^n V$.

so tr $V^n g$ =

$$\sum_{i_1 \le \cdots \le i_n} \alpha_{i_1} \cdots \alpha_{i_n} =$$

$$h_n(\alpha_1, \ldots, \alpha_n) =$$

$$\Delta_{(n)}(\alpha_1, \ldots, \alpha_n)$$

SIMILARLY THERE IS A

n TH EXTERIOR POWER

$$\Lambda^n V = \otimes^n V / \{a - \mathrm{sgn}(\sigma)\, \sigma(a)\}$$

$$a \in \otimes^n V,\ \sigma \in S_n$$

HAS BASIS

$$v_{i_1} \wedge \cdots \wedge v_{i_r}$$

$$i_1 < i_2 < \cdots < i_r.$$

$$\wedge^r V = 0 \text{ IF } r > \text{DIM}(V).$$

$$\text{DIM } \wedge^r V = \binom{n}{r}$$

WE HAVE 2 FAMILIES

OF REPS OF $GL(n, \mathbb{C})$

REPS ON $\vee^r V$, $\wedge^r V$

DIM ARE $\binom{n+k-1}{k}$, $\binom{n}{k}$

CHARS ARE $(x_1, \dots, x_n)$

$$h_k(x_1, \dots, x_n) = S^{(\alpha)}_{(k, 0, \dots, 0)}$$

$$e_k(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_k} x_{i_1} \cdots x_{i_k} = S^{(\alpha)}_{(\underbrace{1, 1, \dots, 1}_{k}, 0, \dots, 0)}$$

BOTH h_k COMPLETE S.P.

e_k ELEMENTARY S.P.

EXAMPLES OF THEOREM RELATING SCHUR POLY TO REPS.

END OF LECTURE

$\otimes^k V$ AND $\vee^k V$ AND $\wedge^2 V$ ARE FUNCTORS. IF $T : V \to W$ A LINEAR MAP THERE IS A MAP

$$\vee^k T : \vee^k V \to \vee^k W$$

$$(\vee^k T)(u_1 \vee \dots \vee u_k) = T(u_1) \vee \dots \vee T(u_k)$$

$$V \xrightarrow{T} W \xrightarrow{S} U$$

$$v^n V \xrightarrow{v^n T} v^n W \xrightarrow{v^n S} v^n U$$

$$v^n(S \circ T) = (v^n S) \circ (v^n T).$$

$$T \in \mathrm{Hom}(V, W)$$

$$v^n T \in \mathrm{Hom}(v^n V, v^n W).$$

$$\mathrm{Hom}(V, W) \to \mathrm{Hom}(v^n V, v^n W)$$

$T : V \to V$

$\vee^k T : \vee^k V \to \vee^k V.$

$\pi : GL(n, \mathbb{C}) \to GL(V)$

STANDARD REP'N ON $V = \mathbb{C}^n$.

USING FUNCTORIALITY GET

$\vee^k \pi(g) : \vee^k V \to \vee^k V$

SO WE GET A REP'N

$\vee^k \pi : GL(n, \mathbb{C}) \to GL(\vee^k V).$

SYMMETRIC POWER REP'N.

$\wedge^k : GL(n, \mathbb{C}) \to GL\left(\binom{n+k-1}{n}, \mathbb{C}\right)$

SIMILAR EXTERIOR POWER REP'N.

FURTHER READING FOR TODAY'S LECTURE:

FULTON AND HARRIS

REPRESENTATION THEORY

FOR TODAY'S LECTURE

APPENDIX B.

IN GENERAL PART III